Maximum Likelihood Estimation for $\mathrm{N}\left(\mu, \sigma^2\right)$

Consider $X_i \overset{\text{iid}}{\sim} \mathrm{N}\left(\mu_0, \sigma^2\right)$ for $i = 1,\ 2, \ldots,\ n$ where the mean, μ_0, is *known*, and the variance, σ^2, is *unknown*. In this setting, $k = 1$ and it is convenient to define $\theta = 1/\sigma^2$. We note that this implies that $\sigma^2 = 1/\theta$.

The joint density function is

$$\begin{aligned} f_{\mathbf{X}}\left(\mathbf{x}|\theta\right) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(x_i - \mu_0\right)^2}{2\sigma^2}\right) \\ &= \left(2\pi\right)^{-n/2} \theta^{n/2} \exp\left(-\frac{\sum_{i=1}^{n}\left(x_i - \mu_0\right)^2}{2}\theta\right) \\ &= \mathrm{L}(\theta|\mathbf{X} = \mathbf{x}) \end{aligned}$$

Noting that $\sum_{i=1}^{n}\left(X_i - \mu_0\right)^2$ is θ-free, we define the statistic $T(\mathbf{X}) = \sum_{i=1}^{n}\left(X_i - \mu_0\right)^2$. This simplifies the above equation to

$$f_{\mathbf{X}}\left(\mathbf{x}|\theta\right) = \left(2\pi\right)^{-n/2} \theta^{n/2} \exp\left(-\frac{T(\mathbf{x})}{2}\theta\right)$$

Letting $C(\theta) = \left(2\pi\right)^{-n/2} \theta^{n/2}$, $h(\mathbf{x}) = 1$, $T(\mathbf{X}) = \sum_{i=1}^{n}\left(X_i - \mu_0\right)^2$, and $\omega(\theta) = -\theta/2$, We see that we have an exponential family of distributions. Hence, $T(\mathbf{X}) = \sum_{i=1}^{n}\left(X_i - \mu_0\right)^2$ is minimally sufficient and complete for $\theta = 1/\sigma^2$, or equivalently $\sigma^2 = 1/\theta$.

So, what is the maximum likelihood estimator of σ^2, $\widehat{\sigma^2} = 1/\widehat{\theta}$? To make life easier, we define the log-likelihood to be

$$\begin{aligned} l(\theta) &= \ln\left(L\left(\theta\right)\right) \\ &= -\frac{n}{2}\ln(2\pi) + \frac{n}{2}\ln(\theta) - \frac{\theta}{2}T \end{aligned}$$

Thus,

$$\dot{l}(\theta) = \frac{d}{d\theta}l(\theta) = \frac{n}{2\theta} - \frac{T}{2}$$

Setting this equal to zero allows us to find $\widehat{\theta}$.

$$\begin{aligned} \frac{n}{2\widehat{\theta}} - \frac{T}{2} = 0 \quad &\Rightarrow \quad \frac{n}{2\widehat{\theta}} = \frac{T}{2} \\ &\Rightarrow \quad \widehat{\theta} = \frac{n}{T} \\ &\Rightarrow \quad \widehat{\sigma^2} = \frac{T}{n} \end{aligned}$$

We need to confirm that $\widehat{\sigma^2}$ maximizes $L\left(\theta\right)$. The second derivative of the log-likelihood is

$$\ddot{l}(\theta) = -\frac{n}{2\theta^2} < 0$$

We do, in fact, have a maximum likelihood estimator, and this estimator is based upon a minimally sufficient and complete statistic, $T(\mathbf{X}) = \sum_{i=1}^{n}\left(X_i - \mu_0\right)^2$. We note that since $T(\mathbf{X})$ is a function of the random variables, $\mathbf{X}$, and a constant, μ_0, and it does not involve the parameter, σ^2, it is a statistic.